Yoneda Ontologies

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Philosophy: Ontology

It's Human nature to categorize things of the same type

As mathematicians, the ways we categorize things are highly expressive.

Throughout history, we have collected:
- **continuity** in topological spaces
- **operations** in groups, etc. algebras
- **directions** in vector spaces
- **structure** in categories

In general, any "expressive form of collecting data", is an **ontology**

Why Categories

An ontology is good for expressing aspects of the data it contains

The question is why did we choose categories to contain theories?

This isn't completely true, categories are eventually insufficient:
- coherance (simplicial sets)
- higher morphisms (n-categories)
- both ((∞, n)-categories)
- varying domains (multicategories)

What are Categories

Among the various forms of ontology, what makes something a "category"?

Is it the notion of morphism and composition?
As formal category theory develops, both notions become stranger

Or is it about how we understand objects via other objects?
i.e. Isomorphism, colimits

Abstractly: We understand objects via their **ambient category**
i.e. the yoneda embedding

Ambient Category

We already understand objects via their ambient categories:

Universal properties:
- understanding complex objects by diagrams of simple objects

Yoneda Lemma:
- understanding objects via the yoneda embedding $Hom(-, a)$

The former motivates **formal ontology**:
Can we describe *formal* objects in an ontology via diagrams?

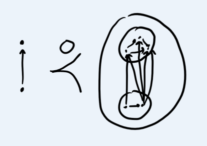
The latter motivates **formal category theory**:
- Yoneda Structures (Street/Walters)
- Virtual Equipments (Shulman/Cruttwell)

Yoneda Lemma v.s. diagrammatic expansions

The yoneda embedding $A \overset{yA}{\to} \mathcal{P}A = [A^{op}, \mathbf{Set}]$ is very well studied

compare to ontological expansions of a graph

$$A \overset{o}{\to} sm(A)$$



both the yoneda embedding and ontological expansions are "ontological transformations"

horizontal morphisms of a "simplicial virtual double category" $\mathbb{O}nt$

Contents

The talk has two* parts:

0: Toy Definition: Basic Ontologies

1: Simplicial virtual double categories

2: Yoneda transformations in sVDCs

Basic Ontologies

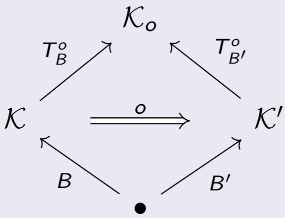
Def: Basic Ontology

A <u>basic ontology</u> is an object of a 2-category $\bullet \xrightarrow{B} \mathcal{K}$

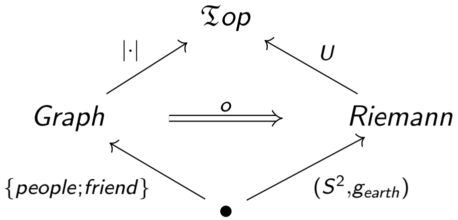
Ex: $\mathcal{K} = \mathbb{C}at$, $s\mathbf{Set}$, $\mathbb{G}\mathbf{Set}$, $\mathfrak{T}op$, $Lax(\Delta, \mathbb{C}at)$

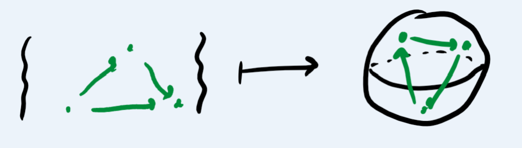
Def: Ontological Transformation

An <u>ontological transformation</u> $B \overset{o}{\nrightarrow} B'$ is a diagram:



Ontological Transformations: Examples





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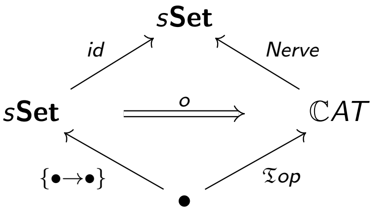
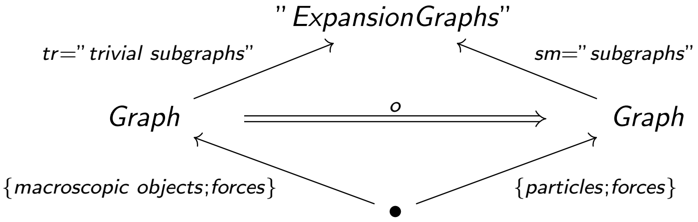
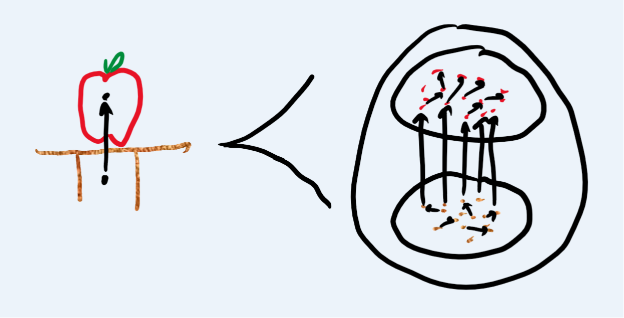


$$\{\bullet \to \bullet\} \overset{o}{\mapsto} \{\mathbb{I} \xrightarrow{x^2} \mathbb{R}\}$$

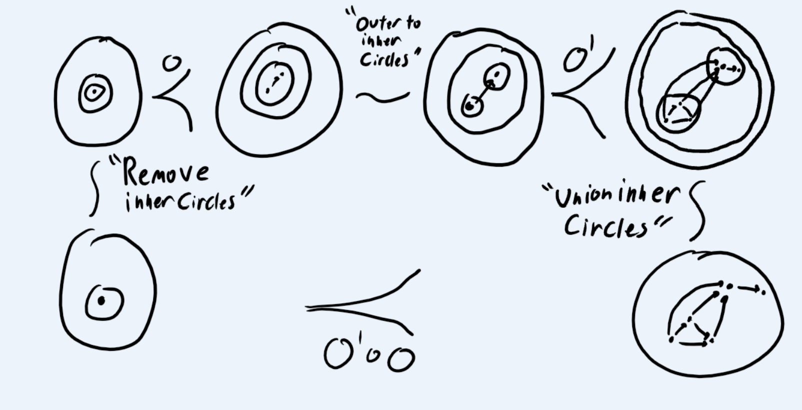
Ontological Transformations: Examples



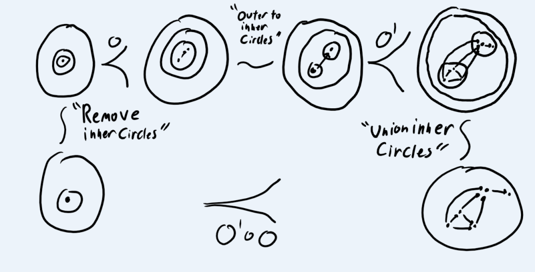


Composition

Given $G \overset{o}{\nrightarrow} G' \overset{o'}{\nrightarrow} G''$, how do we compose?

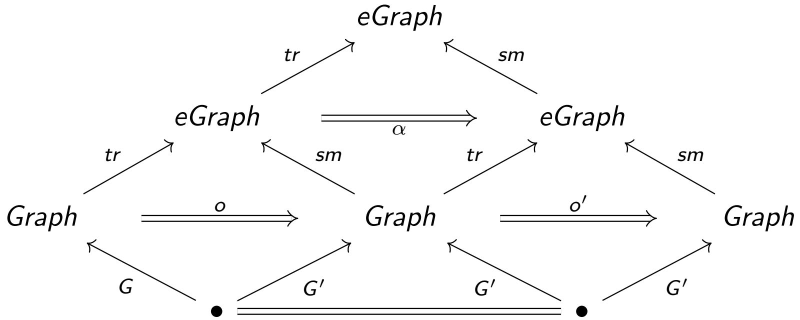


Composition



$$
\begin{array}{ccccccc}
tr(tr(G)) & \xrightarrow{tr(o)} & tr(sm(G')) & \xrightarrow{\alpha_{G'}} & sm(tr(G')) & \xrightarrow{sm(o')} & sm(sm(G'')) \\
\downarrow \alpha_l & & & & & & \downarrow \alpha_r \\
tr(G) & & & \xrightarrow{O' \circ O} & & & sm(G'')
\end{array}
$$

Composition



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Composition

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\downarrow{\scriptstyle \alpha_l} & & & & & & \downarrow{\scriptstyle \alpha_r} \\
tr(G) & & & \xrightarrow{\quad O' \circ O \quad} & & & sm(G'')
\end{array}
$$

if we squint we can package the α's into a single cell

$$
\begin{array}{ccccc}
G & \overset{o}{\longrightarrow} & G' & \overset{o'}{\longrightarrow} & G'' \\
\| & & \Downarrow{\scriptstyle \alpha} & & \| \\
G & & \underset{O' \circ O}{\longrightarrow} & & G'
\end{array}
\qquad \alpha = (\alpha_l, \alpha_G, \alpha_r)
$$

Ontologies form A virtual double category!

… well not quite. Still worth reviewing:

Def: virtual double category

Def: A virtual double category (VDC) $\mathcal{V}$ consists of:

- A "vertical" category $\mathcal{V}_0 = \{A, B; A \xrightarrow{f} B\}$
- A "horizontal" graph $\mathcal{V}_h = \{A, B; A \overset{P}{\nrightarrow} B\}$ with $\mathcal{V}_{h0} = \mathcal{V}_{00}$
- "cells"

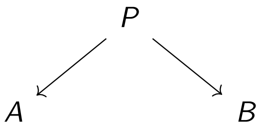
$$\begin{array}{ccccccc} A_0 & \xrightarrow{P_1} & A_1 & \dashrightarrow & A_{n-1} & \xrightarrow{P_n} & A_n \\ {\scriptstyle f}\downarrow & & \Downarrow \alpha & & & & \downarrow{\scriptstyle g} \\ \bullet & & & \xrightarrow[Q]{} & & & \bullet \end{array}$$

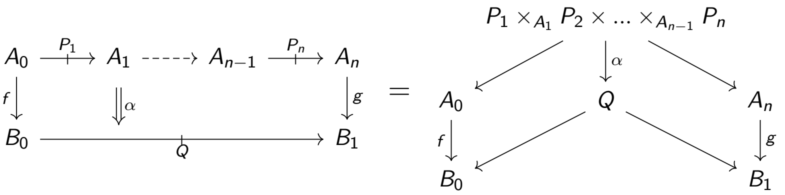
with identities and associative composition of cells

Segway: Span($\mathbb{C}$)

The most relevant example of a VDC
are the spans of a cocomplete category $\mathbb{C}$, $\mathcal{V}$ =span($\mathbb{C}$):

- $\mathcal{V}_0 = \mathbb{C}$
- $A \overset{P}{\nrightarrow} B =$





Simplicial Virtual Double Categories

The problem with treating ontologies as a virtual double category, is akin to trying to construct span($\mathbb{C}$) when $\mathbb{C}$ has no pullbacks.

In the place of a **canonical** "span of spans" (the pullback) we **choose** a span of spans.

The pullback makes span($\mathbb{C}$) a virtual double category.

In the absence of pullbacks, we choose how we compose cells. We should immediately think of simplicial sets.

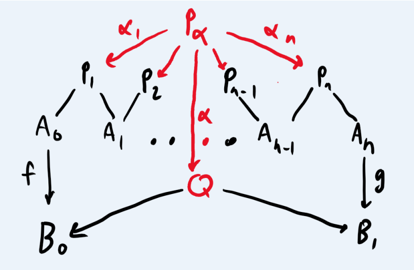
It turns out that span($\mathbb{C}$) and analogously $\mathbb{O}nt$ are simplicial virtual double categories

Segway: Span($\mathbb{C}$)

Span$(\mathbb{C})_{h,1}$:

$$
\begin{array}{ccccccccc}
A_0 & \overset{P_1}{\longrightarrow} & A_1 & \dashrightarrow & A_{n-1} & \overset{P_n}{\longrightarrow} & A_n & & \{P\} \\
{\scriptstyle f}\downarrow & & \Downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle g} & \underset{\sim}{\text{side view}} & \downarrow{\scriptstyle \alpha} \\
B_0 & & & \underset{Q}{\longrightarrow} & & & B_1 & & Q
\end{array}
$$

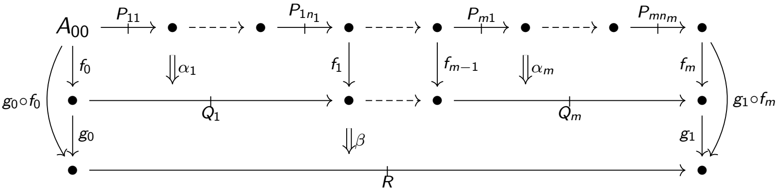
$\curlywedge$



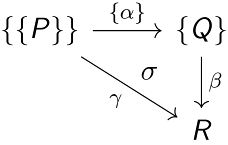
i.e.: $\alpha = (P_\alpha, \alpha_i, \alpha)$

Span($\mathbb{C}$)

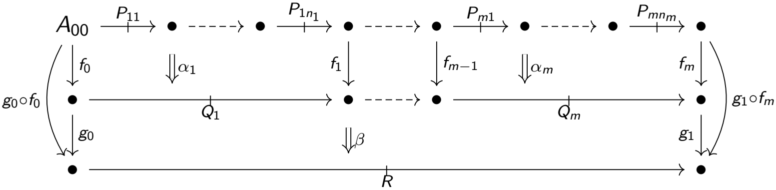
Span$(\mathbb{C})_{h,2}$:



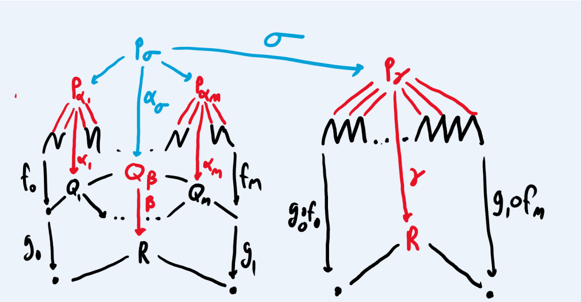
≀ side view



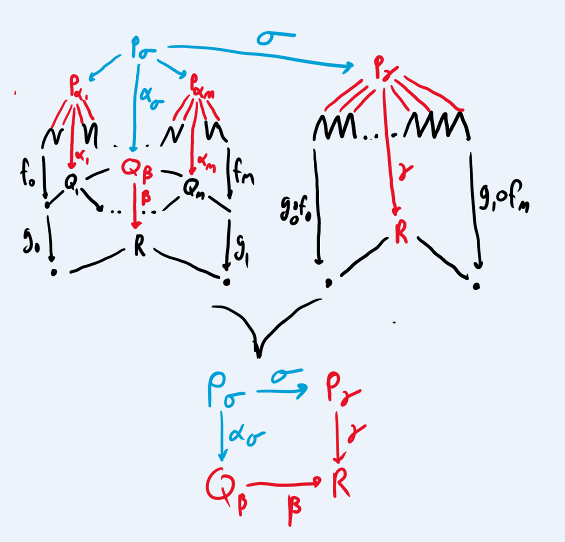
Span(ℂ)



$\curlywedge$

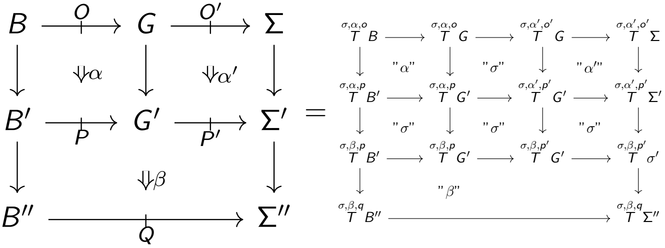


Span(ℂ)

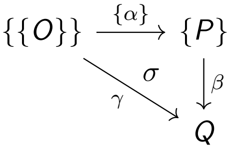


If you think that's complicated...

A simple 2-cell in $\mathbb{O}nt$:



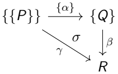
... is rather complicated. But, from the side it's fine :



fc-multicategories

So what is a simplicial virtual double category?

Let's first discuss how to create virtual double categories formally

In the diagram: 

, the {} cooresponds to a monad.

namely, the "free category" or "path" monad:

$$fc = Graph \xrightarrow{Fr} \mathbb{C}at \xrightarrow{U} Graph$$

Originally, VDCs were called fc-Multicategories: [leinster]

fc-Multicategories

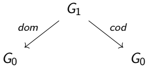
Take a span in Graph:

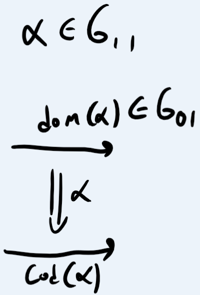
$$
\begin{array}{ccc}
 & G_1 & \\
 {\scriptstyle dom} \swarrow & & \searrow {\scriptstyle cod} \\
G_0 & & G_0
\end{array}
$$

$\alpha \in G_{1,1}$

$\Downarrow \alpha$

fc-Multicategories

Take a span in Graph: 



fc-Multicategories

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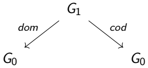
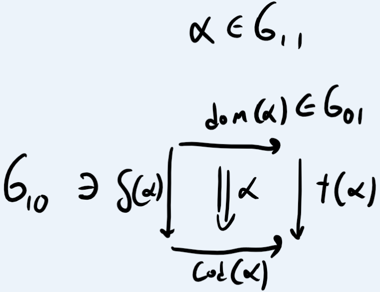
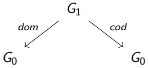
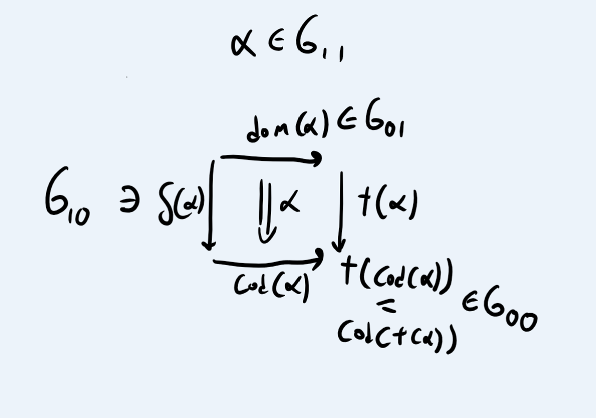




fc-Multicategories

Take a span in Graph: 

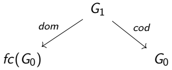


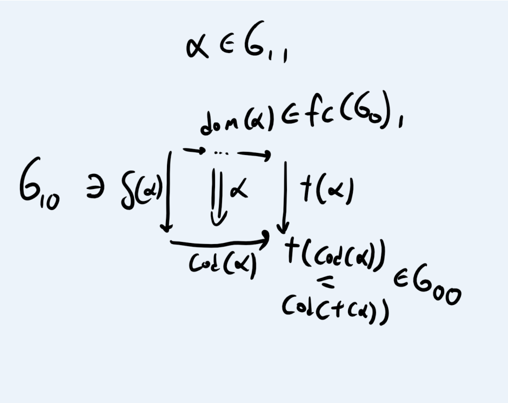
So this corresponds to double categories*

*= just raw data, need to consider "modules" of these spans to get composition

fc-Multicategories

Take a span in Graph:





So this corresponds to virtual double categories*

*= just raw data, need to consider "modules" of these spans to get composition

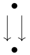
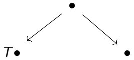
fc-multicategories $\rightarrow$ fc-simplicial sets

This type of span is a member of a horizontal kleisli construction [shullman/cruttwell], i.e.:

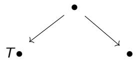
$$\begin{array}{ccc} & G_1 & \\ {}^{dom}\swarrow & & \searrow^{cod} \\ G_0 & & G_0 \end{array} \in \mathit{Span}(\text{Graph})$$

$$\begin{array}{ccc} & G_1 & \\ {}^{dom}\swarrow & & \searrow^{cod} \\ fc(G_0) & & G_0 \end{array} \in \mathbb{H} - kl(\mathit{Span}(\text{Graph}), fc)$$

fc-multicategories $\longrightarrow$ fc-simplicial sets

shape	concept
	internal category (double category)
	kliesli internal category (virtual double category)
Δ	simplicial object (simplicial double set)
???	virtual simplicial double set

KΔ

	kliesli internal category

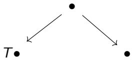
let (T, i, μ) be a monad on $\mathbb{C}$

We will use the same meta-process for $\mathbb{H} - kl(\mathbb{C}, T)$:

- we take copies of the objects in Δ, call them Δ_n^r
- the r flags how many times we will apply a monad T
- we arrange the face maps to mimic the kliesli construction
- we connect everything back together using i and μ

This is what we end up with:

KΔ

	kliesli internal category

$$K\Delta = \left\{ \begin{array}{ccccc} \Delta_n^{r-1} & \xrightarrow{i_n^r} & \Delta_n^r & \xleftarrow{\mu_n^r} & \Delta_n^{r+1} \\ & & f_i \downarrow \uparrow d_i & \searrow^{f_n} & \\ & & \Delta_{n-1}^r & \underset{\mu_{n-1}^r}{\longleftarrow} & \Delta_{n-1}^{r+1} \end{array} \right\}_{(r,n)\in\mathbb{N}\times\mathbb{N}}$$

Like the "ordinary" simplex category except that f_n is flagged for an application of T

K for "kleisli", we will see how to interpret this in a moment

Kleisli Simplicial Objects

Let $\mathbb{C}$ be a category and (T, i, μ) a monad:
$K\Delta(\mathbb{C}, T) = \{\Sigma : K\Delta \to \mathbb{C}$ such that

- $\Sigma(\Delta_n^r) = T^r \Sigma_n$
- $\Sigma(\mu^r) = T^r(\mu)$
- $\Sigma(i^r) = T^r(i)\}$

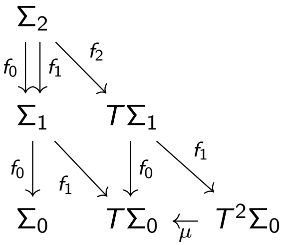
where $\Sigma_n := \Sigma(\Delta_n^0)$

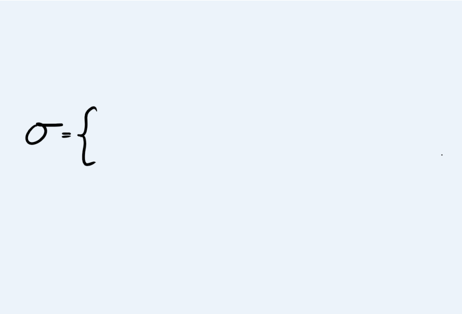
We'll call $\Sigma \in K\Delta(\mathbb{C}, T)$ a **T-simplicial object**

As the simplest example: $K\Delta(\mathbf{Set}, id) = s\mathbf{Set}$

fc - Simplicial graph

So what is $\Sigma \in K\Delta(\text{Graph}, fc)$? Consider $\sigma \in (\Sigma_2)_1$





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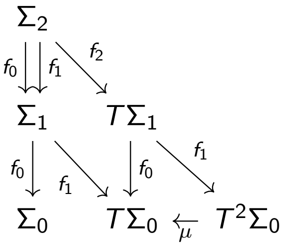

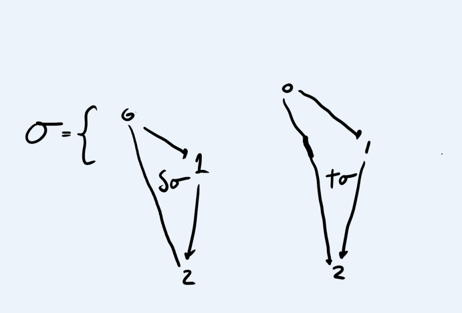

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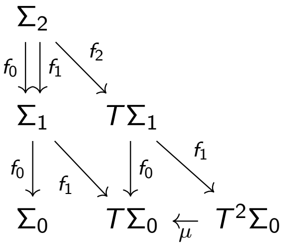

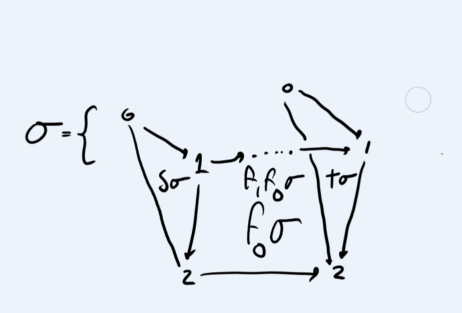

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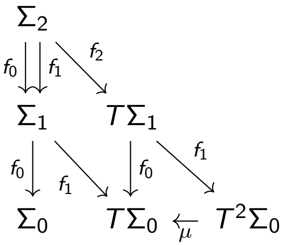

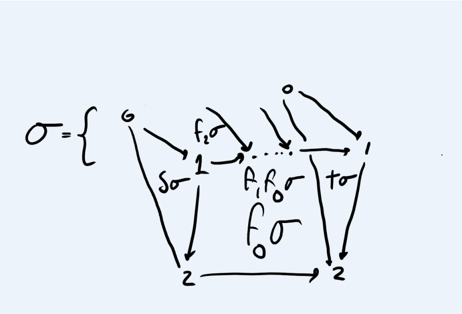

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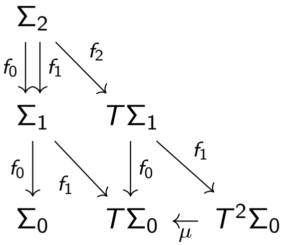
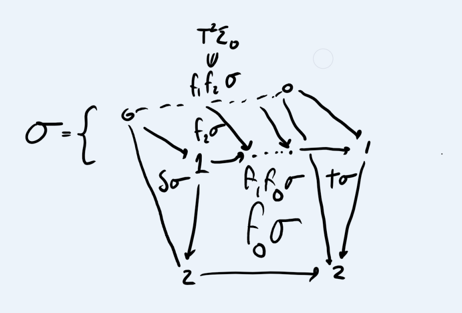
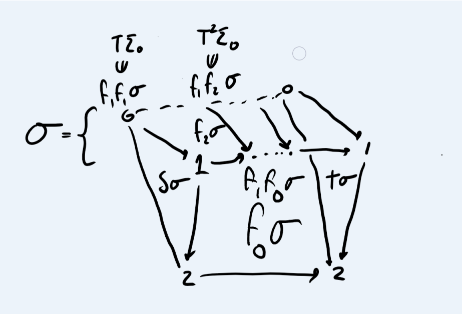




fc - Simplicial graph

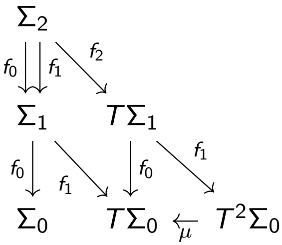
So what is $\Sigma \in K\Delta(\mathrm{Graph}, fc)$? Consider $\sigma \in (\Sigma_2)_1$

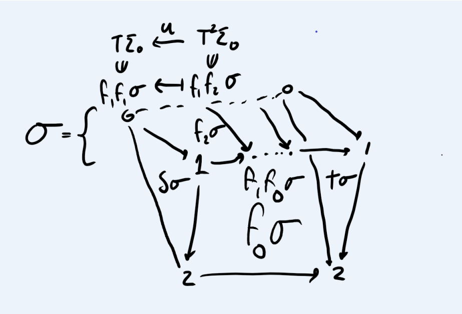
$$
\begin{array}{ccccc}
\Sigma_2 & & & & \\
{\scriptstyle f_0}\downarrow\downarrow{\scriptstyle f_1} & \searrow^{f_2} & & & \\
\Sigma_1 & & T\Sigma_1 & & \\
{\scriptstyle f_0}\downarrow & \searrow^{f_1} & {\scriptstyle f_0}\downarrow & \searrow^{f_1} & \\
\Sigma_0 & & T\Sigma_0 & \xleftarrow[\mu]{} & T^2\Sigma_0
\end{array}
$$



fc - Simplicial graph

So what is $\Sigma \in K\Delta(\text{Graph}, fc)$? Consider $\sigma \in (\Sigma_2)_1$





Simplicial Double Category

Let's rename fc-simplicial graphs
simplicial virtual double sets

Note that for a graph G, $fc(G)_0 = G_0$

thus, $U(\Sigma) = \Sigma_{(-),0}$ is a simplicial set:

$$\begin{array}{ccc} K\Delta & \xrightarrow{\Sigma} & \text{Graph} \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle U} \\ \Delta & \underset{U(\Sigma)}{\dashrightarrow} & \textbf{Set} \end{array}$$

We'll call Σ a **simplicial virtual double category** or **sVDC** if the underlying simplicial set is the nerve of a category.

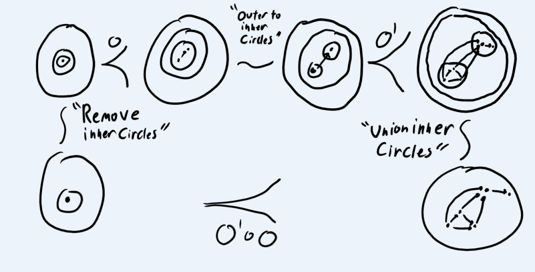
$$U(\Sigma) = N(\mathbb{C})$$

$\mathbb{O}nt$ and Span($\mathbb{C}$)

Span($\mathbb{C}$)is an sVDC
- with underlying category $\mathbb{C}$
- and horizontal 1-cells given by choosing spans of spans

$\mathbb{O}nt$ is an sVDC,
- with underlying category $\{B \in \mathcal{K} \in \mathbb{C}at_2; B \xrightarrow{f} B' \in \mathcal{K} \in \mathbb{C}at_2\}$
- with horizontal 0-cells given by 2-cospans
- and horizontal 1-cells given by 2-cospans of 2-cospans

reminder of cells in Ont



$$
\begin{array}{ccccccc}
tr(tr(G)) & \xrightarrow{tr(o)} & tr(sm(G')) & \xrightarrow{\alpha_{G'}} & sm(tr(G')) & \xrightarrow{sm(o')} & sm(sm(G'')) \\
\downarrow{\scriptstyle \alpha_l} & & & & & & \downarrow{\scriptstyle \alpha_r} \\
tr(G) & & & \xrightarrow{\quad O' \circ O \quad} & & & sm(G'')
\end{array}
$$

Ambient Ontology

We are seeking is a criteria for when an ontology has enough information to describe its objects from the ambient ontology.

when does

$$\text{"points of G"} = tr(G) \to sm(G) = \text{"subgraphs of G"}$$

satisfy similar properties to

$$A \stackrel{yA}{\to} \mathcal{P}A = [A^{op}, Set]$$

the yoneda embedding of [Street / Walters]

Yoneda Lemma

recall the ordinary yoneda lemma:

For any $J : A^{op} \to \mathbf{Set}$,

$$Nat(A(-, a), J) \cong J(a)$$

We can extend this to a parameterized yoneda lemma:

For any $F : A \to B$, and $J : A^{op} \times B \to \mathbf{Set}$,

$$Nat(A(-,-), J(-, F-)) \cong Nat(B(F-,-), J(-,-))$$

Yoneda Lemma

For any $F : A \to B$, and $J : A^{op} \times B \to \mathbf{Set}$,
$Nat(A(-,-), J(-, F-)) \cong Nat(B(F-,-), J(-,-))$

bifunctor	curried
$(J : A^{op} \times B \to \mathbf{Set})$	$(J^{\lambda} : B \to [A^{op}, \mathbf{Set}])$
$(A : A^{op} \times A \to \mathbf{Set})$	$(yA : A \to [A^{op}, \mathbf{Set}])$
$(B(F-,-) : A^{op} \times B \to \mathbf{Set})$	$(B(F, 1) : B \to [A^{op}, \mathbf{Set}])$

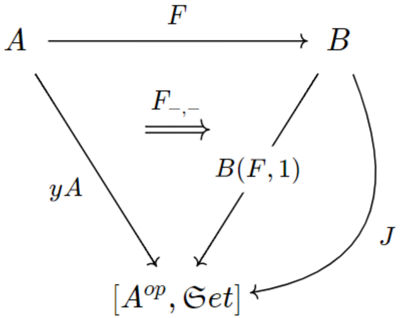
And the yoneda lemma becomes:

$$Nat(yA, J^{\lambda} \circ F) \cong Nat(B(F, 1), J^{\lambda})$$

Yoneda Lemma

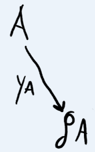
$$Nat(yA, J^\lambda \circ F) \cong Nat(B(F,1), J^\lambda)$$

This is exactly to say that B(F,1) is universal in the sense that it is part of a left extension[Street / Walters]



Yoneda Structures

The basic idea in street and walters, is to replace the 2-category of categories with an arbitrary 2-category $\mathcal{K}$



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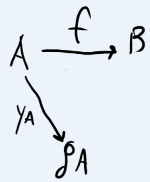

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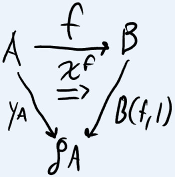

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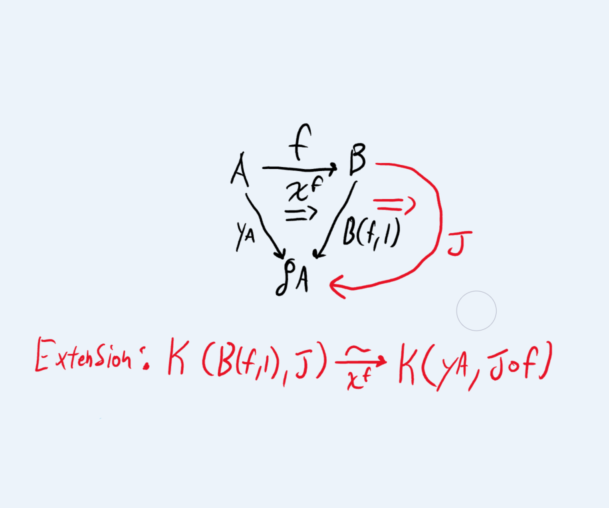

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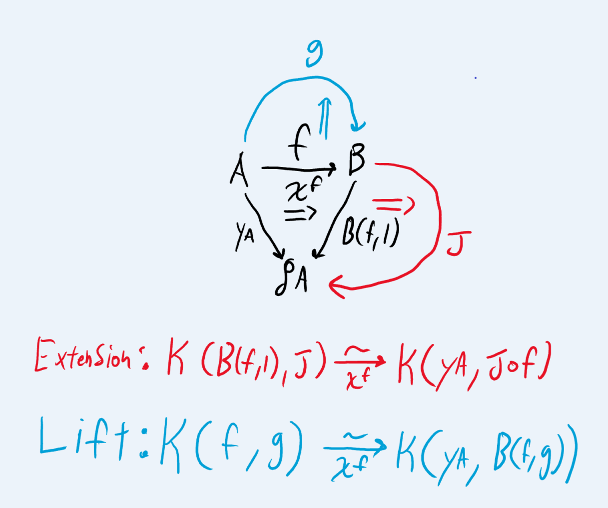


Yoneda Structures in a VDC

we want a "yoneda" criteria for an ontological transformation

$$B \nrightarrow B$$

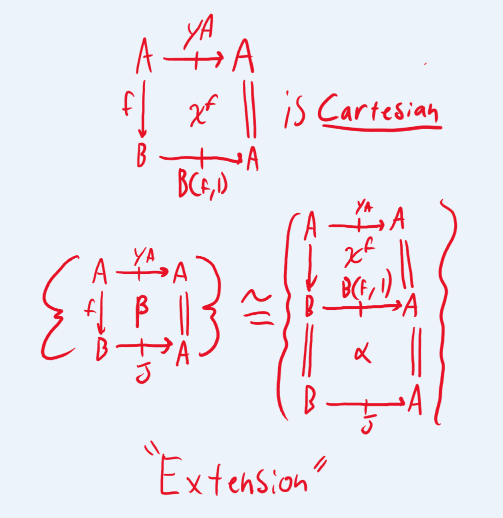
we need to realize the abstract yoneda embedding

$$A \xrightarrow{yA} \mathcal{P}A$$

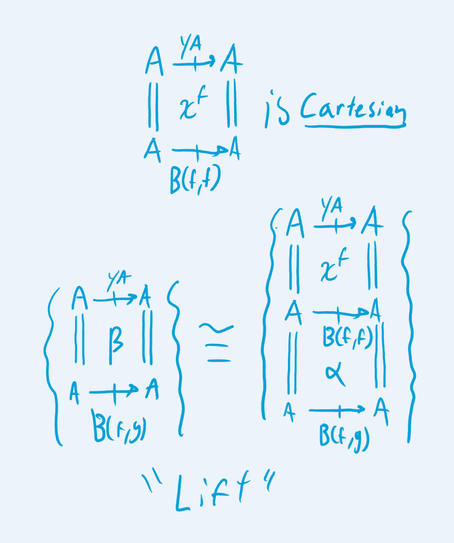
as a horizontal morphism in a VDC (and eventually in an sVDC):

$$A \overset{yA}{\nrightarrow} A$$

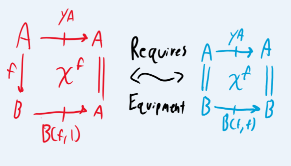
Left Extension in a VDC



Left Lift in a VDC



Yoneda Ontology



Treating these cells as equivalent data requires our VDC to be a **virtual equipment**

Question: what is correct notion of a simplicial virtual equipment?

Def: Yoneda Ontology

A Yoneda Ontology is a basic ontology $\bullet \xrightarrow{B} \mathcal{K}$, with an ontological transformation $B \overset{y}{\nrightarrow} B$ such that for every $A \xrightarrow{f} B \in \mathcal{K}$, we get two cartesian 1-cells as above.

Current Work

Homotopy theory of sVDCs:
- when $\mathbb{C}$ has pullbacks, the ordinary Span($\mathbb{C}$)sits inside of the simplicial Span($\mathbb{C}$)via some nerve construction
- **Question:** what is the nerve for sVDCs?
- What is a quasi-VDC?

Combinatorial Simplification
The higher cells in $\mathbb{O}nt$ demand **painful** combinatorics.
Question: Can use diagrammatic expansion to simplify $\mathbb{O}nt$, the same goes for $Span(\mathrm{C})$.

- Find non-categorical **examples** of yoneda ontologies
i.e. Can we spice up the category of graphs (perhaps adding formal inverses to expansions) to make it a yoneda ontology?

References

VDCs: [Shulman/Crutwell]
https://arxiv.org/abs/0907.2460

Yoneda Structures: [Street/ Walters]
https://www.sciencedirect.com/science/article/pii/0021869378901606

fc-multicategories: [Leinster]
https://arxiv.org/abs/math/0305049

ontological expansions: [Chrein]
https://nchrein.github.io/pages/Talks